Nambu system associated with n-dimensional maps

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abstract

We studied that arbitrary 2-dimensional maps are Hamilton system if a initial value of map is a "time" variable. In this paper, we generalize this correspondence, and show that an n-dimensional map is a Nambu system in which one of initial values of the map play a role of "time" variable.

1 Introduction

In our previous paper [SSYY], we studied 2-dimensional maps :

$$(x_1^i,\ x_2^i) \longmapsto (x_1^{i+1},\ x_2^{i+1})$$

and behaviors of point $(x_1^m(x_1^0,\ x_2^0),\ x_2^m(x_1^0,\ x_2^0))$ maped m times repeatedly. The map is assumed to have its inverse and being differentiable. Changing point of view, let $t \equiv x_1^0$ be a independent "time" variable, $\lambda \equiv x_2^0$ be a fixed parameter, $X(t) \equiv x_1^m$ be a dependent coordinate variable and $P(t) \equiv x_2^m$ be also a dependent momentum variable. We denote by $J^{0,m}$ Jacobi matrix of the map : $(x_1^0, x_2^0) \mapsto (x_1^m, x_2^m)$. In this view-point, we obtained the following result.

Theorem 1 *Let H be a function of (X, P) given by*

$$H(X,P) = \int^{\lambda} (\det J^{0m}) d\lambda, \tag{1}$$

and satisfying

$$\frac{\partial H}{\partial t} = 0.$$

Then the set of Hamilton's equations

$$\frac{dX}{dt} = \frac{\partial H}{\partial P}, \qquad \frac{dP}{dt} = -\frac{\partial H}{\partial X} \tag{2}$$

hold.

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In order to support our claim we derived Hamiltonians corresponding to the Hénon, KdV and $q\mathrm{P}_{\mathrm{IV}}$ maps in [SSYY]. This view-point is based on studies of a discrete version of exact WKB analysis by Shudo and Ikeda.

The aim of this paper is to generalize this mechanism, in order to find a dynamical system associated with n-dimensional map based on this view-point. In consequence, we will show that the corresponding dynamical system is a Nambu system.

Nambu system is a generalized Hamilton dynamical system which was introduced by Nambu, [Na, Ta]. This system is defined by $(n-1)$-Hamiltonians and Nambu brackets which reprace Poisson brackets in the ordinary Hamilton systems. Nambu brackets satisfy some properties such as skew-symmetry, Libnitz rule, fundamental identity and linear combination. Nambu system is useful tool, *ex.* deformation quantization [DFST, DF], dispersionless KP hierarchies and self-dual Einstein equation [Gu], *etc.* And one of the most famous problem is the Euler tops problem which have bi-Hamiltonian structure studied by Nambu [Na]. And more, commutators corresponding to Nambu bracket and algebra of it, called Nambu-Lie algebra, n-Lie algebra, n-ary Lie algebroid or Filippov algebroid are studied in recent [DT, Fi, GM1, GM2, GM3, ILMP, Vai1, Vai2, Val].

2 n-dimensional maps

Let us consider n-dimensional maps and inverse of them:

$$s\ :\ (x_1^i,\ldots,x_n^i)\ \longmapsto\ (x_1^{i+1},\ldots,x_n^{i+1}),\qquad s^{-1}\ :\ (x_1^{i+1},\ldots,x_n^{i+1})\ \longmapsto\ (x_1^i,\ldots,x_n^i),$$
$$x_j^{i+1}:=s(x_j^i)\equiv g_j(x_1^i,\ldots,x_n^i),\qquad x_j^i:=s^{-1}(x_j^{i+1})\equiv g_j^{-1}(x_1^{i+1},\ldots,x_n^{i+1}).$$

where g_j's are some differentiable functions. We consider also Jacobi matrices associated wiht this maps :

$$J^{i,i+1}\ :=\ \left[\frac{\partial x_j^{i+1}}{\partial x_k^i}\right]\ =\ \begin{bmatrix}\dfrac{\partial x_1^{i+1}}{\partial x_1^i} & \cdots & \dfrac{\partial x_1^{i+1}}{\partial x_n^i}\\ \vdots & & \vdots\\ \dfrac{\partial x_n^{i+1}}{\partial x_1^i} & \cdots & \dfrac{\partial x_n^{i+1}}{\partial x_n^i}\end{bmatrix},$$
$$J^{i+1,i}\ :=\ \left[\frac{\partial x_j^{i}}{\partial x_k^{i+1}}\right]\ =\ \begin{bmatrix}\dfrac{\partial x_1^{i}}{\partial x_1^{i+1}} & \cdots & \dfrac{\partial x_1^{i}}{\partial x_n^{i+1}}\\ \vdots & & \vdots\\ \dfrac{\partial x_n^{i}}{\partial x_1^{i+1}} & \cdots & \dfrac{\partial x_n^{i}}{\partial x_n^{i+1}}\end{bmatrix}.$$

The Jacobi matrix $J^{0,m}$ is given by a product of them,

$$J^{0,m}:=J^{0,1}\cdots J^{m-1,m},\qquad J^{m,0}:=J^{m,m-1}\cdots J^{1,0},$$
$$J^{ij}J^{ji}=E,\qquad (E:\ \text{idntity matrix}).$$

If we introduce notations $dx^i = (dx_1^i, \ldots, dx_n^i)^T$ and $\partial^i = (\partial/\partial x_1^i, \ldots, \partial/\partial x_n^i)^T$, the following results hold.

$$dx^{i+1} = J^{i,i+1} dx^i, \qquad dx^i = J^{i+1,i} dx^{i+1}, \tag{3}$$

$$\partial^{i+1} = (J^{i+1,i})^T \partial^i, \qquad \partial^i = (J^{i,i+1})^T \partial^{i+1}, \tag{4}$$

where T express a transposition.

Here, let us change a point of view. We consider a m times repeated map : $x^0 \mapsto x^m$ where x^i is a set of variables $(x_1^i, \ldots, x_n^i)$. We also use notations as follows :

$$(q_1, \ldots, q_n) \equiv (x_1^m, \ldots, x_n^m), \qquad (\lambda_1, \ldots, \lambda_{n-1}, t) \equiv (x_1^0, \ldots, x_n^0)$$

$q_j(t)$ $(j = 1, \ldots, n)$ are coordinates of an n-dimensional phase space, λ_j $(j = 1, \ldots, n-1)$ are fixed parameters and t is a parameter which we consider as an independent "time" variable. In this view-point, the set of variables $q = (q_1, \ldots, q_n)$ satisfy the following dynamical system.

Proposition 1 *Let $h = (h_1, \ldots, h_{n-1})$ be a set of functions of $(q_1(t), \ldots, q_n(t))$ given by*

$$h_i = \int^{\lambda_i} (\det J^{0m})^{\frac{1}{n-1}} d\lambda_i, \qquad i = 1, \ldots, n-1 \tag{5}$$

satisfying

$$\frac{dh_i}{dt} = 0, \qquad i = 1, \ldots, n-1. \tag{6}$$

Then Nambu-Hamilton equations

$$\frac{df}{dt} = \{h_1, \ldots, h_{n-1}, f\} \tag{7}$$

hold, where $f = f(q_1, \ldots, q_n, t)$ is a certain function.

If $f = q_i$ then (7) is an equation of motion. In (7), Nambu brackets is defined by

$$\{f_1, \ldots, f_n\} = \frac{\partial(f_1, \ldots, f_n)}{\partial(q_1, \ldots, q_n)}. \tag{8}$$

Here we assume the existence of the inverse map s^{-1}, such that h_j's are considered as functions of q_j's through $\lambda = \lambda(q) = (s^{-1})^m(q)$.

For simplicity, we define some symbols before proof. H is a Jacobi matrix of $(h_1, \ldots, h_n)$ given by

$$H_q := \left[\frac{\partial h_j}{\partial q_k}\right], \qquad H_\lambda := \left[\frac{\partial h_j}{\partial \lambda_k}\right], \qquad j, k = 1, \ldots, n,$$

and $\tilde{H}_q$ is a cofactor matrix of H_q. Namely the (j,k)-element of $\tilde{H}_q$ is the (j,k)-cofactor of H_q. Here, we set formally $\lambda_n = t$ and

$$h_n := \int^{\lambda_n} (\det J^{0m})^{\frac{1}{n-1}} d\lambda_n$$

Then these matrices satisfy the following Lemma.

Lemma 1 *Let us consider the Nambu-Hamilton equation given by*

$$\frac{\partial f}{\partial \lambda_k} = \{h_1, \ldots, h_{k-1}, f, h_{k+1}, \ldots, h_n\}, \qquad k = 1, \ldots, n, \tag{9}$$

where λ_j $(1 \le j \le n, j \ne k)$ are fixed parameters, λ_k is a independent parameter, q_j $(1 \le j \le n)$ are depnendent parameters $q_j(\lambda_k)$, h_j $(1 \le j \le n)$ are hamiltonians without h_k and f, f_j $(1 \le j \le n)$ are arbitrary functions $f_j(q_1, \ldots, q_n, \lambda_1, \ldots, \lambda_n)$.

Then, for above Jacobi matrices H_q, H_λ, J^{0m}, the cofactor matrices $\bar{H}_q$ and $\bar{H}_\lambda$, following three relations hold.

1. $H_\lambda = H_q J^{0,m}$,
2. $J^{0,m} = \tilde{H}_q^T$,
3. $H_q = (\det \tilde{H}_q)^{\frac{1}{n-1}} (\tilde{H}_q^T)^{-1}$.

Proof of Lemma 1 :
1. $H_\lambda = H_q J^{0,m}$. Using (4),

$$\begin{bmatrix} \partial h_j/\partial \lambda_1 \\ \vdots \\ \partial h_j/\partial \lambda_n \end{bmatrix} = (J^{0,m})^T \begin{bmatrix} \partial h_j/\partial q_1 \\ \vdots \\ \partial h_j/\partial q_n \end{bmatrix}$$

Hence,

$$H_\lambda^T = (J^{0,m})^T H_q^T.$$

Transposing this, therefore, the relation $H_\lambda = H_q J^{0,m}$ hold. ▮

2. $J^{0,m} = \tilde{H}_q^T$. Substituting q_j to f in Nambu-Hamilton equation (9),

$$\frac{\partial q_j}{\partial \lambda_k} = \{h_1, \ldots, h_{k-1}, q_j, h_{k+1}, \ldots, h_n\}$$

Then r.h.s. of this is a (k, j) cofactor, because

$$\frac{\partial(h_1, \ldots, h_{k-1}, q_j, h_{k+1}, \ldots, h_n)}{\partial(q_1, \ldots, , q_n)} = (-1)^{k+j} \frac{\partial(h_1, \ldots, h_{k-1}, h_{k+1}, \ldots, h_n)}{\partial(q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n)} = \tilde{h}_{k,j}.$$

And l.h.s. is one of entries of $J^{0,m}$. Hence we obtain $J^{0,m} = \tilde{H}_q^T$. ▮

3. $H_q = (\det \tilde{H}_q)^{\frac{1}{n-1}} (\tilde{H}_q^T)^{-1}$. It is well known that an arbitrary $n \times n$ matrix A and its cofactor matrix $\tilde{A}$ satisfy the following relation.

$$A\tilde{A}^T = \tilde{A}A^T = (\det A)E.$$

Since this relation derives

$$\det \tilde{A} = (\det A)^{n-1},$$

the matrix A can be expressed by $\tilde{A}$ as follows :

$$A = (\det \tilde{A})^{\frac{1}{n-1}} (\tilde{A}^T)^{-1}.$$

If $A = H_q$, the relation 3 holds. ∎

Proof of Proposition 1 : We assume that maps s, s^{-1} and their explicit forms g_j are given.

(i) : We must show that functions h_j satisfy Nambu-Hamilton equation (7) if h_j are given by (5), because h_j's are given by explicit functions g_j. Hence, we must check a compatibility between (6) and (7). Substituting h_j to (7),

$$\frac{dh_j}{dt} = \{h_1, \ldots, h_{n-1}, h_j\} = 0 \quad \text{if} \quad j \neq n$$

because Nambu bracket is a Jacobian. Therefore if the functions h_j, called Hamiltonians, are given by maps s and s^{-1} with (5), then h_j's satisfy Nambu-Hamilton equation.

(ii) : We will show that if Nambu-Hamilton equation is given by (7) and maps s, s^{-1} are given, then functions h_j are given by (5). Using the three relations of Lemma 1,

$$H_\lambda = H_q J^{0,m} = (\det \tilde{H}_q)^{\frac{1}{n-1}} (\tilde{H}_q^T)^{-1} \tilde{H}_q^T = (\det J^{0,m})^{\frac{1}{n-1}} E.$$

Since r.h.s. is a diagonal matrix, we obtain h_j as follow :

$$\partial_{\lambda_k} h_j = (\det J^{0,m})^{\frac{1}{n-1}} \delta_{j,k} \quad \Longrightarrow \quad h_j = \int^{\lambda_j} (\det J^{0,m})^{\frac{1}{n-1}} d\lambda_j$$

Therefore, if maps and Nambu system are given then Hamiltonian h_j are given by (5).

(iii) : If maps s and s^{-1} has been given, then there exist Nambu system corresponding to maps because of (i) and (ii). The Nambu system have Nambu-Hamilton equation (7) and Hamiltonians (5). ∎

The generalized Nambu-Hamilton equation (9) is not dynamical equation. If we select one independent variable λ_k as a time variable, then Nambu-Hamilton dynamical equation is given by

$$\frac{df}{d\lambda_k} = \{h_1, \ldots, h_{k-1}, f, h_{k+1}, \ldots, h_n\}.$$

This equation is also Nambu-Hamilton equation, and Hamiltonians are h_j, $(j \neq k)$ but h_k is not Hamiltonian. We can choose one independent variable in parameters $(\lambda_1, \ldots, \lambda_n)$ on Nambu system, freely.

On the Nambu system, explicit functions g_j of maps are solutions of Nambu dynamics, because this functions

$$q_j(t) = g_j^m(\lambda_1, \ldots, \lambda_{k-1}, t, \lambda_{k+1}, \ldots, \lambda_n)$$

are depend on $(n-1)$-constants and one independent variable, where g_j^m is a explicit form of m time repeated maps of s.

In the special case of $(\det J^{0m}) = 1$, $(n-1)$-constants are $(n-1)$-Hamiltonians, since

$$h_j = \int^{\lambda_j} d\lambda_j = \lambda_j.$$

And the map s is a canonical transformation or a n-dimensional volume preserving transformation, since

$$dx_1^{i+1} \wedge \cdots \wedge dx_n^{i+1} = dx_1^i \wedge \cdots \wedge dx_n^i,$$

and

$$dq_1 \wedge \cdots \wedge dq_n = d\lambda_1 \wedge \cdots \wedge dt \wedge \cdots \wedge d\lambda_n = dh_1 \wedge \cdots \wedge dt \wedge \cdots \wedge dh_n.$$

So, (h_j, t) is a set of canonical conjugate variables.

In our sense, a independent "time" value is a initial value. This mean that the response of changes of a initial value in discrete systems can be investigated with Nambu mechanics in continuum systems because of this Nambu-map correspondence.

3 Example

3.1 Lotka-Volterra map

Discrete Lotka-Volterra equation :

$$\bar{x}_k\,(1 + \bar{x}_{k-1}) = x_k(1 + x_{k+1}), \qquad k = 1, 2, 3$$

have a 3-dimensional map and its inverse

$$\bar{x}_k = x_k \frac{1 + x_{k+1} + x_{k+1}x_{k+2}}{1 + x_{k+2} + x_{k+2}x_k}, \qquad x_k = \bar{x}_k \frac{1 + \bar{x}_{k+2} + \bar{x}_{k+2}\bar{x}_{k+1}}{1 + \bar{x}_{k+1} + \bar{x}_{k+1}\bar{x}_k}$$

under periodic boundary condition $x_{k+3} = x_k$, where $\bar{x}_k = x_k^{i+1}$, $x_k = x_k^i$. Jacobi matrix $J^{i,i+1}$ is given by

$$\begin{bmatrix} \dfrac{(1+x_2+x_2x_3)(1+x_3)}{(1+x_3+x_3x_1)^2} & -\dfrac{x_2(1+x_2+x_2x_3)}{(1+x_1+x_1x_2)^2} & \dfrac{x_3(1+x_2)}{1+x_2+x_2x_3} \\ \dfrac{x_1(1+x_3)}{1+x_3+x_3x_1} & \dfrac{(1+x_3+x_3x_1)(1+x_1)}{(1+x_1+x_1x_2)^2} & -\dfrac{x_3(1+x_3+x_3x_1)}{(1+x_2+x_2x_3)^2} \\ -\dfrac{x_1(1+x_1+x_1x_2)}{(1+x_3+x_3x_1)^2} & \dfrac{x_2(1+x_1)}{1+x_1+x_1x_2} & \dfrac{(1+x_1+x_1x_2)(1+x_2)}{(1+x_2+x_2x_3)^2} \end{bmatrix}$$

and its Jacobian and inverse are the following

$$\det J^{i,i+1} = 1, \qquad \det J^{i+1,i} = 1$$

because $J^{i,i+1}J^{i+1,i} = E$. Now, we will consider the simplest case $m = 1$. Setting up variables as follows,

$$(h_1, h_2, t) = (\lambda_1, \lambda_2, \lambda_3) = (x_1^0, x_2^0, x_3^0), \qquad (q_1, q_2, q_3) = (x_1^1, x_2^1, x_3^1),$$

satisfy the following Nambu system.

- Equations of motion

$$\frac{dq_1}{dt} = \frac{\partial(h_1, h_2)}{\partial(q_2, q_3)}, \qquad \frac{dq_2}{dt} = -\frac{\partial(h_1, h_2)}{\partial(q_1, q_3)}, \qquad \frac{dq_3}{dt} = \frac{\partial(h_1, h_2)}{\partial(q_1, q_2)},$$

- Hamiltonians

$$h_1 = q_1 \frac{1 + q_3 + q_3 q_2}{1 + q_2 + q_2 q_1}, \qquad h_2 = q_2 \frac{1 + q_1 + q_1 q_3}{1 + q_3 + q_3 q_2},$$

- Solutions

$$q_1(t) = h_1 \frac{1 + h_2 + h_2 t}{1 + t + h_1 t}, \qquad q_2(t) = h_2 \frac{1 + t + h_1 t}{1 + h_1 + h_1 t}, \qquad q_3(t) = t \frac{1 + h_1 + h_1 h_2}{1 + h_2 + h_2 t},$$

- Explicit forms of equations of motion

$$\frac{dq_1}{dt} = \frac{-q_1(1 + q_1 + q_1 q_3)}{(1 + q_2 + q_2 q_1)(1 + q_3 + q_3 q_2)},$$
$$\frac{dq_2}{dt} = \frac{q_2(1 + q_2)(1 + q_1 + q_1 q_3)}{(1 + q_2 + q_2 q_1)(1 + q_3 + q_3 q_2)},$$
$$\frac{dq_3}{dt} = \frac{(1 + q_3)(1 + q_1 + q_1 q_3)}{(1 + q_2 + q_2 q_1)(1 + q_3 + q_3 q_2)}.$$

References

[DFST] G. Dito, M. Flato, D. Sternheimer, L. Takhtajan, *Deformation Quantization and Nambu Mechanics*, Commun. Math. Phys. **183** (1997) 1-22, preprint hep-th/9602016.

[DF] G. Dito, M. Flato, *Generalized Abelian Deformations: Application to Nambu Mechanics*, Lett. Math. Phys. **39** (1997) 107-125, hep-th/9609114

[DT] Y. L. Daletskii, L. Takhtajan, *Leibniz and Lie algebra structures for Nambu algebra*, Lett. Math. Phys. **39** (1997), 127–141.

[DZ] J.-P. Dufour and M. Zhitomirskii, *Nambu structures and integrable 1-forms*, math.DG/0002167

[Fi] V. T. Filippov, *n-Lie algebras*, Sibirsk. Math. Zh. **26**(6) (1985), 126–140.

[GM1] J. Grabowski, and G. Marmo, *Remarks on Nambu-Poisson and Nambu-Jacobi brackets*, J. Phys. A: Math. Gen. **32** (1999), 4239–4247, preprint math.DG/9902128.

[GM2] J. Grabowski, and G. Marmo, *On Filippov algebroids and multiplicative Nambu-Poisson structures*, Diff. Geom. Appl. **12** (2000), 35–50, preprint math.DG/9902127.

[GM3] J. Grabowski, G. Marmo, *Non-antisymmetric versions of Nambu-Poisson and Lie algebroids*, J. Phys. A: Math. Gen. **34** (2001), 3803-3809, preprint math.DG/0104122.

[Gu] P. Guha, *Volume preserving multidimensional integrable systems and Nambu-Poisson Geometry*, math-ph/9807018

[ILMP] R. Ibanez, M. de Leon, J. C. Marrero and E. Padron, *Leibniz algebroid associated with a Nambu-Poisson structure*, math-ph/9906027

[M] P. Monnier, *Computations of Nambu-Poisson cohomologies*, math.DG/0007103

[Na] Y. Nambu, *Generalized Hamiltonian mechanics*, Phys. Rev. **D7** (1973), 2405–2412.

[SSYY] S. Saito, A. Shudo, J. Yamamoto and K. Yoshida, *Hamilton flow associated maps* nlin.SI/0108018.

[Ta] L. Takhtajan, *On foundation of the generalized Nambu mechanics*, Commun. Math. Phys. **160** (1994), 295–315, preprint hep-th/9301111.

[Vai1] I. Vaisman, *Nambu-Lie Groups*, math.DG/9812064

[Vai2] I. Vaisman, *A Survey on Nambu-Poisson Brackets*, math.DG/9901047

[Val] J. A. Vallejo, *Nambu-Poisson manifolds and associated n-ary Lie algebroids*, J. Phys. A: Math. Gen. **34** (2001) 2867-2881, preprint math-ph/0104040

[Var] P. Varga, *A Modification of Nambu's Mechanics*, quant-ph/9701006